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Paths and Edge-Connectivity in Graphs

HARUKO OKAMURA

Faculty of Engineering, Osaka City University, Osaka 558, Japan

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Mader proved that for every k-edge-connected graph G ($k \geqslant 4$), there exists a path joining two given vertices such that the subgraph obtained from G by deleting the edges of the path is $(k-2)$-edge-connected. A generalization of this and a sufficient condition for existance of 3, 4, or 5 terminus k edge-disjoint paths in graphs are given. © 1984 Academic Press, Inc.

1. INTRODUCTION

We consider finite undirected graphs passibly with multiple edges but without loops. Let G be a graph and let $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. For two distinct vertices x and y, let $\lambda_G(x, y)$ be the maximal number of edge-disjoint paths between x and y, and let $\lambda_G(x, x) = \infty$. For an integer $k \geqslant 1$, let $\Gamma(G, k)$ be

$$\{X \subseteq V(G) \mid \text{for each } x, y \in X, \lambda_G(x, y) \geqslant k\}.$$

Let $(s_1, t_1),..., (s_k, t_k)$ be pairs of vertices of G. When is the following statement true?

(1.1) *There exist edge-disjoint paths $P_1,..., P_k$ such that P_i has ends s_i, t_i $(1 \leqslant i \leqslant k)$.*

Seymour [7] and Thomassen [8] characterised such graphs when $k = 2$, and Seymour [7] when $|\{s_1,..., s_k, t_1,..., t_k\}| = 3$.

For integers $k \geqslant 1$ and $n \geqslant 2$, set

$$g(k) = \min\{m \mid \text{if } G \text{ is } m\text{-edge-connected, then (1.1) holds}\},$$

$$\lambda'(k, n) = \min\{m \mid \text{if } |\{s_1,..., s_k, t_1,..., t_k\}| \leqslant n \text{ and } \{s_1,..., s_k, t_1,..., t_k\} \in \Gamma(G, m), \text{then (1.1) holds}\},$$

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$$\lambda(k, n) = \min\{m \mid \text{if } |\{s_1,..., s_k, t_1,..., t_k\}| \leqslant n \text{ and } \lambda_G(s_i, t_i) \geqslant m \ (1 \leqslant i \leqslant k), \text{then (1.1) holds}\},$$

and set

$$\lambda'(k) = \lambda'(k, 2k) = \lambda'(k, m) \ (m > 2k) \qquad \text{and} \qquad \lambda(k) = \lambda(k, 2k).$$

Then for each $k \geqslant 1$,

$$\lambda'(k, 3) = \lambda(k, 3) \qquad \text{and} \qquad \lambda(k) \geqslant \lambda'(k) \geqslant g(k) \geqslant k.$$

For $n \geqslant 4$ and even integer $k \geqslant 2$,

$$g(k) > k \qquad \text{and} \qquad \lambda(k) \geqslant \lambda(k, n) \geqslant \lambda'(k, n) > k$$

(see Fig. 1 in which $k/2$ represents the number of parallel edges).

Thomassen [8] gave Conjecture 1, and we give Conjecture 2 slightly stronger than Conjecture 1.

CONJECTURE 1. *For each integer* $k \geqslant 1$,

$$g(k) = \begin{cases} k & \textit{if } k \textit{ is odd}, \\ k+1 & \textit{if } k \textit{ is even}. \end{cases}$$

CONJECTURE 2. *For each integer* $k \geqslant 1$,

$$\lambda(k) = \begin{cases} k & \textit{if } k \textit{ is odd}, \\ k+1 & \textit{if } k \textit{ is even}. \end{cases}$$

Clearly $\lambda(1) = 1$. Cypher [1] proved $\lambda(2) = 3$ and $\lambda(k) \leqslant k+2$ ($k = 3, 4, 5$), and $\lambda(3) = 3$ was announced in [5] and proved in [6] by the author. Enomoto and Saito [2] proved $g(4) = 5$, and independently Hirata, Kubota, and Saito [3] proved $\lambda(4) = 5$ and $\lambda(k) \leqslant 2k-3$ ($k \geqslant 6$).

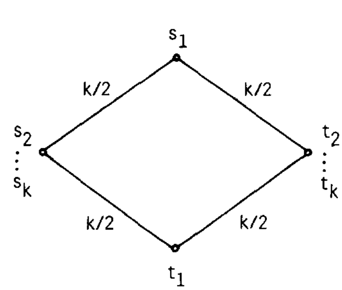

FIGURE 1

The following theorems are useful when we consider the edge-disjoint paths problem.

THEOREM 1. *Suppose that $k \geqslant 4$ is an integer, G is a graph, $\{s, t\} \subseteq T \subseteq V(G)$ and $T \in \Gamma(G, k)$. Then*

(1) *For each nonseparating edge e incident to s, there exists a path P between s and t passing through e such that*

$$T \in \Gamma(G - E(P), k-2) \qquad \text{and} \qquad \{s, t\} \in \Gamma(G - E(P), k-1).$$

(2) *For each vertex a of $T - \{s, t\}$ of degree less than $2k$ and for each edge f incident to a, there exists a path P between s and t not passing through a such that*

$$T \in \Gamma(G - E(P), k-2), \qquad \{s, t, a\} \in \Gamma(G - E(P), k-1),$$

and

$$\{s, a\} \qquad \text{or} \qquad \{t, a\} \in \Gamma(G - E(P) - f, k-1).$$

(3) *For each vertex a with $\lambda_G(s, a) < k$, there exists a path P between s and t not passing through a such that*

$$T \in \Gamma(G - E(P), k-2), \qquad \{s, t\} \in \Gamma(G - E(P), k-1),$$

and for $x = s, t$,

$$\lambda_{G-E(P)}(x, a) = \lambda_G(x, a).$$

(4) *For each nonseparating edges e and e' incident to s, there exists a cycle C passing through e and e' such that*

$$T \in \Gamma(G - E(C), k-2).$$

(Here $G - E(P)$ denotes the subgraph obtained from G by deleting the edges of P.)

COROLLARY 1. *For every k-edge-connected graph G $(k \geqslant 4)$ and for every vertices x, y of G, there exists a path P between x and y such that $G - E(P)$ is $(k-2)$-edge-connected.*

Theorem 1 is a generalization of an unpublished result of Mader given in Corollary 1. Since $\lambda(3) = 3$, from Corollary 1 it follows that $g(4) = 5$.

THEOREM 2. *Suppose that $k \geqslant 4$ and $n \geqslant 2$ are integers, G is a graph and $T = \{s_1, \dots, s_n, t_1, \dots, t_n\} \subseteq V(G)$. If $T \notin \Gamma(G, k)$ and for each $1 \leqslant i \leqslant n$,*

$$\lambda_G(s_i, t_i) \geqslant k,$$

then for some $1 \leqslant j < l \leqslant n$, there exist disjoint paths P_1 between s_j and t_j and P_2 between s_l and t_l such that

$$\{s_i, t_i\} \in \Gamma\left(G - \bigcup_{i=1}^{2} E(P_i), k-2\right) \qquad (1 \leqslant i \leqslant n).$$

THEOREM 3. *Suppose that $n \geqslant 4$ is an integer and $k \geqslant 3$ is an odd integer. If for each odd integer $1 \leqslant m \leqslant k$,*

$$\lambda'(m, n) = m,$$

then

$$\lambda(k, n) = k \qquad \text{and} \qquad \lambda(k+1, n) = k+2.$$

From Theorem 3 it follows that $\lambda(4) = 5$.

THEOREM 4. *Suppose that $k \geqslant 2$ is an integer, G is a graph, $\{a_1, a_2\} \subseteq T \subseteq V(G)$, $|T| \leqslant 3$ and $T \in \Gamma(G, k)$. Then there exists a path P between a_1 and a_2 such that $T \in \Gamma(G - E(P), k-1)$.*

THEOREM 5. *Suppose that $k \geqslant 3$ is an odd integer, G is a graph, $\{a_1, a_2, a_3\} \subseteq T \subseteq V(G)$, $a_2 \neq a_3$ and $T \in \Gamma(G, k)$. Then*

(1) *If $|T| \leqslant 4$, then there exists a path P between a_1 and a_2 such that $T \in \Gamma(G - E(P), k-1)$.*

(2) *For $m = 2, 3$ if $|T| \leqslant 4$ and for $m = 3$ if $|T| = 5$ and $k \geqslant 5$, there exist edge-disjoint paths P_1 between a_1 and a_2 and P_2 between a_1 and a_m such that $T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-2)$.*

THEOREM 6. *For each integer $k \geqslant 1$,*

$$\lambda(k, 3) = k \qquad \text{and} \qquad \lambda(k, 4) = \lambda(k, 5) = \begin{cases} k & \text{if } k \text{ is odd}, \\ k+1 & \text{if } k \text{ is even}. \end{cases}$$

In Theorem 5(2) if $m = 2$ and $|T| = 5$, then the conclusion does not always hold. Figure 2 gives a counterexample with $k = 7$.

When k is odd and $|\{s_1, \dots, s_k, t_1, \dots, t_k\}| \geqslant 4$, if for some $1 \leqslant i \leqslant k$,

$$\lambda_G(s_i, t_i) < k,$$

then (1.1) does not always hold. Figure 3 gives a counterexample.

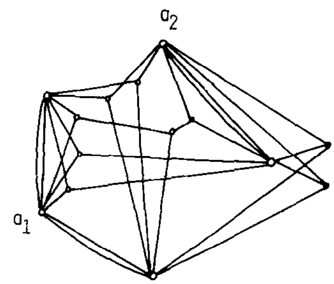

FIGURE 2

Notations and Definitions

Let $X, Y \subseteq V(G)$, $F \subseteq E(G)$, $\{x, y\} \subseteq V(G)$, and $e \in E(G)$. We often denote $\{x\}$ by x and $\{e\}$ by e. The subgraph of G induced by X is denoted by $\langle X \rangle_G$ and the subgraph obtained from G by deleting X (F) is denoted by $G - X$ ($G - F$). We denote by $\partial_G(X, Y)$ the set of edges with one end in X and the other in Y, and $\partial_G(X)$ denotes $\partial_G(X, V(G) - X)$. We denote by $\lambda_G(X, Y)$ the maximal number of edge-disjoint paths with one end in X and the other in Y. We call $\partial_G(X)$ an n-cut if $|\partial_G(X)| = n$ and $\langle X \rangle_G$ and $\langle V(G) - X \rangle_G$ are both connected. An n-cut $\partial_G(X)$ is called nontrivial if $|X| \geqslant 2$ and $|V(G) - X| \geqslant 2$, trivial otherwise. We denote by $d_G(x)$ the degree of x and $N_G(x)$ denotes the set of vertices adjacent to x. We regard a path and a cycle as subgraphs of G. A path $P = P[x, y]$ denotes a path between x and y, and for $x', y' \in V(P)$, $P(x', y')$ denotes a subpath of P between x' and y'.

2. Proof of Theorem 1

For a vertex $w \in V(G)$ and $b, c \in N_G(w)$, we let $G_w^{b,c}$ be the graph $(V(G), (E(G) \cup e) - \{f, g\})$, where e is a new edge with ends b and c, $f \in \partial_G(w, b)$ and $g \in \partial_G(w, c)$. We require

LEMMA 2.1 (Mader [4]). *Suppose that w is a nonseparating vertex of a*

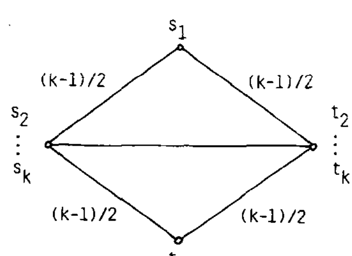

FIGURE 3

graph G with $d_G(w) \geqslant 4$ and with $|N_G(w)| \geqslant 2$. Then there exist $b, c \in N_G(w)$ such that for each $x, y \in V(G) - w$,

$$\lambda_{G_w^{b,c}}(x, y) = \lambda_G(x, y).$$

We prove Theorem 1 by induction on $|E(G)|$. If $|T| = 1$, then $s = t$ and the results holds, and so we may assume that $|T| \geqslant 2$ and $s \neq t$. If G is not 2-connected, then we can deduce the results by using induction on some blocks. Thus we may assume that G is 2-connected.

Case 1. G has a nontrivial k-cut $\partial_G(X) = \{e_1, ..., e_k\}$ $(X \subseteq V(G))$ separating T.

Let H (K) be the graph obtained from G by contracting $V(G) - X$ (X) to a new vertex u (v). Set $T_H = (X \cap T) \cup u$ and $T_K = (T - X) \cup v$. Let $s \in X$. Note that $\{e_1, ..., e_k\}$ is contained in $E(H)$ and also in $E(K)$.

(1) Let $t \in X$. By induction H has a path $P[s, t]$ such that $e \in E(P)$, $T_H \in \Gamma(H - E(P), k-2)$, and $\{s, t\} \in \Gamma(H - E(P), k-1)$. If $u \notin V(P)$, then P is a required path of G. If $u \in V(P)$, then we may let $\{e_1, e_2\} \subseteq E(P)$. By induction K has a cycle C such that $\{e_1, e_2\} \subseteq E(C)$ and $T_K \in \Gamma(K - E(C), k-2)$. Now we can construct a required path of G. Let $t \in V(G) - X$. H has a path $P_1[s, u]$ such that $e \in E(P_1)$, $T_H \in \Gamma(H - E(P_1), k-2)$ and $\{s, u\} \in \Gamma(H - E(P_1), k-1)$. We may let $e_1 \in E(P_1)$. K has a path $P_2[v, t]$ such that $e_1 \in E(P_2)$, $T_K \in \Gamma(K - E(P_2), k-2)$ and $\{v, t\} \in \Gamma(K - E(P_2), k-1)$. Now we can construct a required path of G.

(2) and (3). If $\{a, t\} \subseteq X$, $a \in X$ and $t \in V(G) - X$, or $\{a, t\} \subseteq V(G) - X$, then we can deduce the results similarly to (1). Let $a \in V(G) - X$ and $t \in X$. By induction for each $1 \leqslant i \leqslant k$, H has a path $P_i[s, t]$ such that $u \notin V(P_i)$, $T_H \in \Gamma(H - E(P_i), k-2)$, $\{s, t, u\} \in \Gamma(H - E(P_i), k-1)$ and for $x = s$ or t, $\{x, u\} \in \Gamma(H - E(P_i) - e_i, k-1)$ (say $x = t$ for $i = 1$).

Let $a \in T$. K has a path $P[a, v]$ such that $f \in E(P)$, $T_K \in \Gamma(K - E(P), k-2)$ and $\{a, v\} \in \Gamma(K - E(P), k-1)$. We may let $e_1 \in E(P)$. Since

$$\lambda_{H - E(P_1) - e_1}(t, u) = k - 1 = \lambda_{K - \{e_1, f\}}(v, a),$$

we have

$$\lambda_{G - E(P_1) - f}(t, a) = k - 1,$$

and so P_1 is a required path of G.

Let $\lambda_G(s, a) < k$. For some $1 \leqslant i \leqslant k$ (say for $i = 1$),

$$\lambda_{K - e_i}(v, a) = \lambda_K(v, a) = \lambda_G(t, a) = \lambda_G(s, a).$$

Since

$$\lambda_{H-E(P_1)-e_1}(t, u) = k-1 \qquad \text{and} \qquad \lambda_{K-e_1}(v, a) = \lambda_G(t, a),$$

we have

$$\lambda_{G-E(P_1)}(t, a) = \lambda_G(t, a).$$

Then

$$\lambda_{G-E(P_1)}(s, a) \geqslant \min\{\lambda_{G-E(P_1)}(s, t), \lambda_{G-E(P_1)}(t, a)\} = \lambda_G(s, a),$$

and so P_1 is a required path of G.

(4) Similar to (1).

Case 2. In (3), G has a nontrivial $\lambda_G(s, a) - \text{cut}\, \partial_G(X)$ $(X \subseteq V(G))$ separating s and a.

Let $s \in X$ and $a \in V(G) - X$. Since $\lambda_G(s, a) < k$, $T \subseteq X$. Let H be the graph obtained from G by contracting $V(G) - X$ to a. Then by induction (3) holds in H, and so in G.

Case 3. Case 1 or 2 does not occur.

Let T_1 be T for (1), (2), and (4) and $T \cup a$ for (3). If an edge g of G is not incident to any vertex of T_1, then we can apply induction on $G - g$. Thus we may assume that each edge is incident to a vertex of T_1. Let $x \in V(G) - T_1$ if such an x exists. If $d_G(x) \geqslant 4$, then by Lemma 2.1 there exist $b, c \in N_G(x)$ such that for each $y, z \in V(G) - x$,

$$\lambda_{G_x^{b,c}}(y, z) = \lambda_G(y, z).$$

By induction the results hold in $G_x^{b,c}$, thus we may let $d_G(x) = 3$. If $|N_G(x)| = 2$, then for some $y \in T$, $|\partial_G(x, y)| = 2$ and for $h \in \partial_G(x, y)$ with $h \neq e$, we can apply induction on $G - h$. Thus we may let $|N_G(x)| = 3$.

Assume first that $|T| = 2$. Then $V(G) = T$ for (1), (2), and (4), and so the results follows. For (3)

$$d_G(a) = |\partial_G(a, s)| + |\partial_G(a, t)| + |V(G) - T_1|$$

and

$$d_G(s) = |\partial_G(s, a)| + |\partial_G(s, t)| + |V(G) - T_1|.$$

Since $d_G(a) < k \leqslant d_G(s)$, we have

$$|\partial_G(s, t)| > |\partial_G(a, t)| \geqslant 0.$$

Thus the result easily follows.

Let $|T| \geqslant 3$.

(1) Let $w \in T - \{s, t\}$. By Lemma 2.1 there exist $b, c \in N_G(w)$ such that for each $x, y \in V(G)$,

$$\lambda_{G_w^{b,c}}(x, y) = \lambda_G(x, y).$$

Set $G' = G_w^{b,c}$ and $T' = T - w$. By induction G' has a path $P'[s, t]$ such that $e \in E(P')$, $T' \in \Gamma(G' - E(P'), k-2)$ and $\{s, t\} \in \Gamma(G' - E(P'), k-1)$. Let P_1 be the corresponding path in G.

$$T - w \in \Gamma(G - E(P_1), k-2) \quad \text{and} \quad \{s, t\} \in \Gamma(G - E(P_1), k-1).$$

For a path P of G, let $A(P)$ be

$$\{x \mid x \in V(P) \cap N_G(w).\ E(P) \cap \partial_G(w, x) \neq \varnothing \text{ or } x \notin T.\}.$$

Let $|A(P_1)| \leqslant 2$. Then in $G - E(P_1)$ there exist $k-2$ edges $g_1, \ldots, g_{k-2}$ incident to w such that the other end of g_i is in T or adjacent to a vertex of $T - w$ $(1 \leqslant i \leqslant k-2)$. Thus

$$\lambda_{G - E(P_1)}(w, T - w) \geqslant k - 2.$$

Hence $T \in \Gamma(G - E(P_1), k-2)$, and P_1 is a required path. If $|A(P_1)| \geqslant 3$, then starting at s along P_1, let x_1 and x_2 be the first and the last vertices of $A(P_1)$, respectively. Let P_2 be the path obtained by combining $P_1(s, x_1)$, g_1, g_2 and $P_1(x_2, t)$, where $g_i \in \partial_G(w, x_i)$ $(i = 1, 2)$. Then for each $y, z \in V(G)$,

$$\lambda_{G - E(P_2)}(y, z) \geqslant \lambda_{G - E(P_1)}(y, z).$$

Moreover $|A(P_2)| = 2$. Thus P_2 is a required path.

(2) Let $|T| = 3$. We may let $T = \{s, t, a\}$. If for some $y \in V(G) - T$, $\partial_G(a, y) = \{f\}$, then the path $P[s, t]$ with $E(P) \subseteq \partial_G(y)$ is a required path. We may let $f \in \partial_G(a, x)$ for $x = s$ or t, say $x = s$. If $\partial_G(s, t) \neq \varnothing$, then a path $P[s, t]$ with $|E(P)| = 1$ is a required path. If $\partial_G(s, t) = \varnothing$, then $|V(G)| > |T|$, because $d_G(a) < 2k$ and $\lambda_G(s, t) \geqslant k$;

$$\lambda_G(a, t) = \lambda_{G-f}(a, t),$$

and so for some $y \in V(G) - T$, the path $P[s, t]$ with $E(P) \subseteq \partial_G(y)$ is a required path. If $|T| \geqslant 4$, then we choose $w \in T - \{s, t, a\}$ and we can deduce the result similarly as (1).

(3) For some $w \in T - \{s, t\}$, we define G' and T' similarly as in (1). Then G' has a path $P'[s, t]$ such that $a \notin V(P')$, $T \in \Gamma(G - E(P'), k-2)$, $\{s, t\} \in \Gamma(G - E(P'), k-1)$ and for $x = s, t$, $\lambda_{G - E(P')}(x, a) = \lambda_G(x, a)$. Let P_1 be the path of G corresponding to P'. We define $A(P_1)$ similarly as in (1).

Then we may assume $A(P_1) \leqslant 2$ (see the proof of (1)). If $\partial_G(w, a) = \varnothing$, then the result follows. Let $\partial_G(w, a) \neq \varnothing$. Since

$$|\partial_G(w) - \partial_G(w, a)| + \min(|\partial_G(a) - \partial_G(a, w)|, |\partial_G(a, w)|) \geqslant k$$

and

$$\lambda_{G-E(P_1)}(s, a) = \lambda_G(s, a) = d_G(a),$$

we have

$$\lambda_{G-E(P_1)}(w, T-w) \geqslant k-2.$$

Now the result follows.

(4) Similar to (1).

3. Proof of Theorem 2

Lemma 3.1. *Suppose that $k \geqslant 4$ and $n \geqslant 1$ are integers, G is a graph, $T = \{s_1, \dots, s_n, t_1, \dots, t_n\} \subseteq V(G)$, $\lambda_G(s_i, t_i) \geqslant k$ $(1 \leqslant i \leqslant n)$, $a \in V(G)$, and $d_G(a) < k$. If for each $X \subseteq V(G)$ such that $\partial_G(X)$ separates $T \cup a$, $|\partial_G(X)| \geqslant d_G(a)$, then for some $1 \leqslant j \leqslant n$, there exists a path $P[s_j, t_j]$ such that $\{s_i, t_i\} \in \Gamma(G - E(P), k-2)$ $(1 \leqslant i \leqslant n)$ and $\lambda_{G-E(P)}(s_j, a) = d_G(a)$.*

Proof. We proceed by induction on $|E(G)|$. If $T \in \Gamma(G, k)$, then from Theorem 1 the result follows, and so we may assume that for some $X \subseteq V(G)$, $\partial_G(X)$ separates T and $|\partial_G(X)| < k$. Choose X with this property such that $|\partial_G(X)|$ is minimum. We may assume that $a \in V(G) - X$ and $T \cap X = \{s_1, \dots, s_m, t_1, \dots, t_m\}$. Let H be the graph obtained from G by contracting $V(G) - X$ to a new vertex u. By induction for some $1 \leqslant j \leqslant m$, H has a path $P[s_j, t_j]$ such that $\{s_i, t_i\} \in \Gamma(H - E(P), k-2)$ $(1 \leqslant i \leqslant m)$ and $\lambda_{H-E(P)}(s_j, u) = d_H(u)$. It easily follows that $\{s_i, t_i\} \in \Gamma(G - E(P), k-2)$ $(1 \leqslant i \leqslant n)$ and $\lambda_{G-E(P)}(s_j, a) = d_G(a)$, and so Lemma 3.1 is proved.

Now we prove Theorem 2. Since $T \notin \Gamma(G, k)$, for some $X \subseteq V(G)$, $\partial_G(X)$ separates T and $|\partial_G(X)| < k$. Choose X with this property such that $|\partial_G(X)|$ is minimum. We may assume that

$$T \cap X = \{s_1, \dots, s_m, t_1, \dots, t_m\} \qquad \text{and} \qquad T - X = \{s_{m+1}, \dots, s_n, t_{m+1}, \dots, t_n\}.$$

Let H (K) be the graph obtained from G by contracting $V(G) - X$ (X) to a new vertex u (v). By Lemma 3.1 for some $1 \leqslant j \leqslant m$, H has a path $P_1[s_j, t_j]$ such that $\{s_i, t_i\} \in \Gamma(H - E(P_1), k-2)$ $(1 \leqslant i \leqslant m)$ and $\lambda_{H-E(P_1)}(s_j, u) = d_H(u)$, and for some $m+1 \leqslant l \leqslant n$, K has a path $P_2[s_l, t_l]$ such that $\{s_i, t_i\} \in$

$\Gamma(K-E(P_2), k-2)$ $(m+1 \leqslant i \leqslant n)$ and $\lambda_{K-E(P_2)}(s_l, v) = d_K(v)$. Now it easily follows that

$$\{s_i, t_i\} \in \Gamma\left(G - \bigcup_{i=1}^{2} E(P_i), k-2\right) \qquad (1 \leqslant i \leqslant n),$$

and so Theorem 2 is proved.

4. Proof of Theorem 3

For each odd integer $1 \leqslant m \leqslant k$, since $\lambda'(m, n) = m$, by Theorem 1 it follows that $\lambda'(m+1, n) = m+2$. Let $\alpha = \varnothing$ or 1 and $\beta = 2\alpha$. We prove $\lambda(k+\alpha, n) = k+\beta$ by induction on k. We may assume $k+\alpha \geqslant 4$. Suppose that G is a graph, $T = \{s_1, ..., s_{k+\alpha}, t_1, ..., t_{k+\alpha}\} \subseteq V(G)$, $|T| \leqslant n$ and $\lambda_G(s_i, t_i) \geqslant k+\beta$ $(1 \leqslant i \leqslant k+\alpha)$. We prove that for $k+\alpha$ instead of k, (1.1) holds in G. Then Theorem 3 is proved. Since $\lambda'(k+\alpha, n) = k+\beta$, we may assume that $T \notin \Gamma(G, k)$. Then by Theorem 2 for some $1 \leqslant j < l \leqslant k+\alpha$, there exist disjoint paths $P_1[s_j, t_j]$ and $P_2[s_l, t_l]$ such that $\{s_i, t_i\} \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k+\beta-2)$ $(1 \leqslant i \leqslant k+\alpha)$. By induction $\lambda(k+\alpha-2, n) = k+\beta-2$. Hence $G - \bigcup_{i=1}^{2} E(P_i)$ has edge-disjoint paths $P_3[s_3, t_3], ..., P_{k+\alpha}[s_{k+\alpha}, t_{k+\alpha}]$, and so the result follows.

5. Proof of Theorem 4

We proceed by induction on $|E(G)|$. We may let $a_1 \neq a_2$ and $|T| = 3$. If G has a nontrivial k-cut $\partial_G(X)$ $(X \subseteq V(G))$ separating T, then we define H, K, u, and v similarly as in the proof of Theorem 1. We may let $|T \cap X| = 2$. By induction for H and $(T \cap X) \cup u$ instead of for G and T, the result holds. Thus the result follows. Hence we may assume that each edge is incident to a vertex of T. If there exists $x \in V(G) - T$, then we may assume that $d_G(x) = 3$ and $N_G(x) = T$ (see the proof of Theorem 1), and so the path $P[a_1, a_2]$ with $E(P) \subseteq \partial_G(x)$ is a required path. If $V(G) = T$, then the result easily follows.

6. Proof of Theorem 5

We call a graph G elemental for $V_1 \subseteq V(G)$ if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$ and for each $x \in V_2$, $d_G(x) = 3$, $|N_G(x)| = 3$ and $N_G(x) \subseteq V_1$. We call a graph G elemental for $V_1 \subseteq V(G)$ and an integer $k \geqslant 1$ if G is elemental for V_1 and for each $x \in V_1$, $d_G(x) = k$. For integers $p \geqslant 0$ and $q \geqslant 0$, we say that a graph G is $G(p, q)$ if G is elemental for some $V_1 =$

$\{x_1, x_2, x_3\} \subseteq V(G)$, $|V(G) - V_1| = q$, and $|\partial_G(x_i, x_j)| = p$ $(1 \leqslant i < j \leqslant 3)$. Let G be an elemental graph for $V_1 \subseteq V(G)$. We call a subgraph S an elemental star if $V(S) \subseteq V_1$, $|V(S)| = 2$, and $|E(S)| = 1$, or if for some $x \in V(G) - V_1$, $V(S) = N_G(x) \cup x$, and $E(S) = \partial_G(x)$.

We require the following lemmas.

LEMMA 6.1. *Suppose that $k \geqslant 3$ is an integer, G is an elemental graph for $T \subseteq V(G)$ and k, $T \in \Gamma(G, k)$, G has no nontrivial k-cut separating T, and that S_1, S_2, S_3 are elemental stars of G. If $V(S_1) \cap V(S_2) \cap V(S_3) = \varnothing$, then $T \in \Gamma(G - \bigcup_{i=1}^{3} E(S_i), k-2)$.*

Proof. Assume that $X \subseteq V(G)$, $|X| \leqslant |V(G) - X|$, and X separates T. Set $G' = G - \bigcup_{i=1}^{3} E(S_i)$. If $|X| = 1$, then let $X = \{x\}$. Since $d_{G'}(x) \geqslant d_G(x) - 2 = k - 2$, we have $|\partial_{G'}(X)| \geqslant k - 2$. If $|X| \geqslant 2$, then $|\partial_G(X)| \geqslant k + 1$, and so $|\partial_{G'}(X)| \geqslant k - 2$. Now Lemma 6.1 is proved.

LEMMA 6.2. *Suppose that $k \geqslant 2$ is an integer, G is an elemental graph for $T = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ and k, $|T| = 4$ and $T \in \Gamma(G, k)$. Then*

(1) *One of the following holds*:

(i) $\partial_G(x_1, x_2) \neq \varnothing$, $\partial_G(x_1, x_3) \neq \varnothing$, *or for some* $y \in V(G) - T$, $N_G(y) = \{x_1, x_2, x_3\}$.

(ii) *k is even*, $|\partial_G(x_2, x_3)| = k/2$, *and*

$$|\{y \in V(G) - T \mid N_G(y) = \{x_i, x_1, x_4\}\}| = k/2 \qquad (i = 2, 3).$$

(2) *One of the following holds*:

(i) *For each $1 \leqslant i < j \leqslant k$, G has an elemental star S containing x_i and x_j.*

(ii) *k is even and G is the graph obtained from four cycle by replacing each edge by $k/2$ parallel edges.*

(3) *If G has no nontrivial k-cut separating T, then*

(i) *$\partial_G(x_1, x_2) \neq \varnothing$ or G has two elemental stars containing x_1 and x_2.*

(ii) *One of the following holds.*

(a) *G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_3]$ such that for $i = 2$ or 4,*

$$\{x_i, x_3\} \in \Gamma\left(G - \bigcup_{j=1}^{2} E(P_j), k-1\right) \quad \text{and} \quad T \in \Gamma\left(G - \bigcup_{j=1}^{2} E(P_j), k-2\right).$$

(b) *For each $e \in \partial_G(x_3) - \partial_G(x_3, x_2)$, G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_3]$ such that $e \in E(P_2)$ and $T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-2)$.*

Proof. For $1 \leqslant i, j \leqslant 4$, set

$$p_{i,j} = |\partial_G(x_i, x_j)|,$$
$$R_i = \{ y \in V(G) - T \mid N_G(y) = T - x_i \},$$
$$r_i = |R_i|.$$

(1) Assume $p_{1,2} = p_{1,3} = r_4 = 0$. Then

$$d_G(x_1) = k = p_{1,4} + r_2 + r_3,$$
$$d_G(x_4) = k = p_{1,4} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3.$$

Thus

$$p_{2,4} = p_{3,4} = r_1 = 0.$$

Since $T \in \Gamma(G, k)$, we have

$$|\partial_G(\{x_2, x_3\})| = r_2 + r_3 \geqslant k.$$

Thus

$$p_{1,4} = 0.$$

By comparing $d_G(x_i)$ with $d_G(x_j)$ for $1 \leqslant i < j \leqslant 3$, we have

$$r_2 = r_3 = p_{2,3}.$$

Now (ii) follows.

(2) Assume $p_{1,2} = r_3 = r_4 = 0$. Then by comparing $d_G(x_1) + d_G(x_2)$ with $d_G(x_3) + d_G(x_4)$, we have

$$r_1 = r_2 = p_{3,4} = 0.$$

Now by comparing $d_G(x_3) = k = p_{1,3} + p_{2,3}$ with $d_G(x_i)$ for $i = 1, 2$, we have

$$p_{1,4} = p_{2,3} \qquad \text{and} \qquad p_{2,4} = p_{1,3}.$$

Moreover,

$$|\partial_G(\{x_1, x_4\})| = p_{1,3} + p_{2,4} = 2p_{1,3} \geqslant k,$$
$$|\partial_G(\{x_1, x_3\})| = p_{1,4} + p_{2,3} = 2p_{1,4} \geqslant k.$$

Thus

$$p_{1,3} = p_{2,3} = p_{2,4} = p_{1,4},$$

and so (ii) follows.

(3)(i) We assume $p_{1,2} = r_4 = 0$, and then prove $r_3 \geqslant 2$. Since any cut separating $\{x_1, x_3\}$ and $\{x_2, x_4\}$ or separating $\{x_1, x_4\}$ and $\{x_2, x_3\}$ has more than k edges, we have

(6.1) $\quad p_{1,4} + p_{2,3} + p_{3,4} + r_1 + r_2 + r_3 \geqslant k + 1,$

and

(6.2) $\quad p_{1,3} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3 \geqslant k + 1.$

By comparing $d_G(x_3) + d_G(x_4)$ with (6.1) and (6.2), we have

$$r_3 \geqslant 2.$$

(ii) If there exists an $f \in \partial_G(x_1, x_3)$, then by Theorem 1, G has a path $P[x_3, x_2]$ such that $f \in E(P)$, $\{x_3, x_2\} \in \Gamma(G - E(P), k-1)$ and $T \in \Gamma(G - E(P), k-2)$, and so (a) follows. Thus we may let

$$p_{1,3} = p_{1,2} = 0,$$

then by (1)

$$r_4 > 0.$$

If $r_3 > 0$, then for $y_1 \in R_4$ and $y_2 \in R_3$,

$$\{x_3, x_4\} \in \Gamma\left(G - \bigcup_{i=1}^{2} \partial_G(y_i), k-1\right) \quad \text{and} \quad T \in \Gamma\left(G - \bigcup_{i=1}^{2} \partial_G(y_i), k-2\right),$$

and so (a) follows. Thus we may let

$$r_3 = 0.$$

Then by (1) and (3)

$$p_{1,4} > 0 \qquad \text{and} \qquad r_4 \geqslant 2.$$

Let y be another end of e, then $y = x_4$ or $y \in R_i$ ($i = 1$, 2 or 4). In each case (b) easily follows.

LEMMA 6.3. *Suppose that $k \geqslant 3$ is an odd integer, G is a graph, $\{x_1, x_2, x_3\} \subseteq T \subseteq V(G)$, $x_i \neq x_j$ $(1 \leqslant i < j \leqslant 3)$, $T \in \Gamma(G, k)$ and $e \in E(G)$. If one of* (i) *or* (ii) *below holds, then for $m = 2, 3$, G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_m]$ such that $e \in E(P_1) \cup E(P_2)$ and $T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-2)$.*

(i) $e \in \partial_G(x_1, x_2)$,

(ii) *$e \in \partial_G(x_1, y)$ for some $y \in V(G) - T$ with $d_G(y) = 3$ and with $N_G(y) = \{x_1, x_2, x_3\}$.*

Proof. Assume that (i) holds. By Theorem 1 if $m = 2$, then G has a cycle C such that $e \in E(C)$ and $T \in \Gamma(G - E(C), k-2)$, and if $m = 3$, then G has a path $P[x_2, x_3]$ such that $e \in E(P)$ and $T \in \Gamma(G - E(P), k-2)$.

Assume that (ii) holds. We may assume that G is 2-connected. If $d_G(x_3) = d > k$, then we replace x_3 by d vertices of degree k (Fig. 4 gives an example with $d = 8$ and $k = 5$), producing a new graph G'. In G' we assign x_3 on $N_{G'}(y) - \{x_1, x_2\}$. If the result holds in G', then clearly the result holds in G, and so we may assume that $d_G(x_3) = k$. Let $f \in \partial_G(x_3) - \partial_G(y, x_3)$. By Theorem 1 G has a path $P[x_1, x_2]$ such that $x_3 \notin V(P)$, $T \in \Gamma(G - E(P), k-2)$, $\{x_1, x_2, x_3\} \in \Gamma(G - E(P), k-1)$ and $\{x_i, x_3\} \in \Gamma(G - E(P) - f, k-1)$ ($i = 1$ or 2). Then $y \notin V(P)$, because $d_G(x_3) = k$ and $d_G(y) = 3$. Moreover, $T \in \Gamma(G - E(P) - y, k-2)$. Thus the result follows.

Now we prove Theorem 5. We may assume that G is 2-connected, $d_G(x) = k$ for each $x \in T$ (see the proof of Lemma 6.3 and Fig. 4, in this case we can assign x on any vertex of new $d_G(x)$ vertices of degree k) and that $d_G(y) = 3$ for each $y \in V(G) - T$ (see Case 3 in the proof of Theorem 1). We proceed by induction on $|E(G)|$. If $|T| \leqslant 3$, then the results follow from Theorem 4. Thus let $|T| \geqslant 4$.

Case 1. G has a nontrivial k-cut $\partial_G(X) = \{e_1, \ldots, e_k\}$ $(X \subseteq V(G))$ separating T.

We define H, K, u, v, T_H, and T_K similarly as Case 1 in the proof of Theorem 1. If $|X \cap T| = 1$, then the results hold in K, and so in G. Thus let $|X \cap T| \geqslant 2$ and $|T - X| \geqslant 2$.

We require the following:

(6.3) *If G has a nontrivial k-cut $\partial_G(Y) = \{f_1, \ldots, f_k\}$ $(Y \subseteq X)$ separating T, then we may assume that $(X - Y) \cap T \neq \varnothing$.*

Proof. Assume $(X - Y) \cap T = \varnothing$. Let b_i (c_i) be the end of e_i (f_i) in $V(G) - X$ (Y) $(1 \leqslant i \leqslant k)$. We may assume that the graph obtained from $\langle X - Y \rangle_G$ by adding $b_1, \ldots, b_k, c_1, \ldots, c_k, e_1, \ldots, e_k, f_1, \ldots, f_k$ has edge-disjoint paths $P_1[b_1, c_1], \ldots, P_k[b_k, c_k]$. Let G' be the graph obtained from $G - (X - Y)$ by adding new edges $g_1, \ldots, g_k$, where g_i has ends b_i and c_i

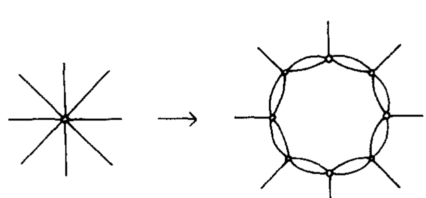

FIGURE 4

$(1 \leqslant i \leqslant k)$. Then $|E(G')| < |E(G)|$, and the results of Theorem 5 hold in G', and so in G. Now (6.3) is proved.

(6.4) *If* $|X \cap T| = 2$ $(|T - X| = 2)$, *then we may assume that* H (K) *is* $G(p, q)$ $(G(p', q'))$ *for some integers* p *and* q $(p'$ *and* $q')$.

Proof. Assume $|X \cap T| = 2$. If H has a nontrivial k-cut $\partial_H(Y)$ $(Y \subseteq V(H) - u)$ separating T_H, then by (6.3) $(X - Y) \cap T \neq \emptyset$, and so $|T \cap Y| = 1$. Then by taking Y instead of X the results of Theorem 5 hold. Thus we may assume that an end of each edge of H is in T_H. Hence the result easily follows.

We return to the proof of Theorem 5. By Lemma 6.3 we may assume the following.

(6.5) $\partial_G(a_1, a_i) = \emptyset$ $(i = 2, m)$ *and for each* $y \in V(G) - T$, $\{a_1, a_2, a_m\} \not\subseteq N_G(y)$.

Let $a_1 \in X$.

(1) Now $|X \cap T| = |T - X| = 2$. If $a_2 \in X$, then by (6.4) the result easily follows. Thus let $a_2 \in V(G) - X$. Since

$$p + q \geqslant (k+1)/2 \qquad \text{and} \qquad p' + q' \geqslant (k+1)/2,$$

for some $1 \leqslant i \leqslant k$, H has an elemental star S_1 containing a_1 and e_i and K has an elemental star S_2 containing a_2 and e_i. Then $T \in \Gamma(G - \bigcup_{i=1}^{2} E(S_i), k-1)$.

(2) Subcase 1.1. $\{a_2, a_m\} \subseteq X$.

H has required paths. If one of them passes through u, then we can deduce the result by using Theorem 1(4) on K.

Subcase 1.2. $\{a_2, a_m\} \subseteq V(G) - X$ and $|X \cap T| = 2$.

Set $X \cap T = \{a_1, a_5\}$. If $|T| = 4$, then a_4 does not exist. By (6.4) H is $G(p, q)$. Thus if one of (6.6) or (6.7) below holds, then the result follows.

(6.6) *For some* $e_i \in \partial_H(u, a_1)$, K *has edge-disjoint paths* $P_1[v, a_2]$ *and* $P_2[v, a_m]$ *such that* $e_i \in E(P_1) \cup E(P_2)$ *and* $T_K \in \Gamma(K - \bigcup_{i=1}^{2} E(P_i), k-2)$.

(6.7) *For some* $e_i, e_j \in \partial_H(u) - \partial_H(u, a_5)$, K *has edge-disjoint paths* $P_1[v, a_2]$ *and* $P_2[v, a_m]$ *such that* $\{e_i, e_j\} \subseteq E(P_1) \cup E(P_2)$ *and* $T_K \in \Gamma(K - \bigcup_{i=1}^{2} E(P_i), k-2)$.

If $p = 0$, then $\partial_H(u, a_5) = \emptyset$, and so (6.7) follows. Thus let $p > 0$. If $|T - X| = 2$, then by (6.4) K is $G(p', q')$, and so (6.6) follows. Thus let $|T - X| = 3$ and $m = 3$. Set $T - X = \{a_2, a_3, a_4\}$.

Subcase 1.2.1. K has a nontrivial k-cut $\partial_K(Y)$ $(Y \subseteq V(K) - v)$ separating T_K.

By (6.3) we may let $|Y \cap T_K| = |T_K - Y| = 2$. Let K_1 and K_2 be the graphs obtained from K by contracting Y and $V(K) - Y$ to a vertex respectively. Then similarly as (6.4) K_i is $G(p_i, q_i)$ for some integers p_i and q_i $(i = 1, 2)$. Let M be

$$\{\{x_1, x_2\} \subseteq V(K) - T_k \mid \partial_K(x_1, x_2) \neq \emptyset\},$$

and let M' be

$$\{x \mid \text{for some } N \in M,\ x \in N\}.$$

For each $N \in M$, $N \cap V(K_i) \neq \emptyset$ $(i = 1, 2)$,

$$d_{K-N}(a_j) = d_{K-N}(v) = k - 1 \quad (j = 2, 3, 4) \qquad \text{and} \qquad T_K \in \Gamma(K - N, k - 1).$$

If $k = |M|$, then $p_1 = p_2 = 0$ and the result easily follows, and so let $k > |M|$. $K - M'$ is elemental for T_K and $k - |M|$.

Assume that $k - |M|$ is even and $K - M'$ is the graph obtained from four cycle by replacing each edge by $(k - |M|)/2$ parallel edges. For each cycle C of $K - M'$ such that $|V(C)| = |E(C)| = 4$, we have $T_K \in \Gamma(G - E(C), k - 2)$. If $\partial_G(a_1, a_4) \neq \emptyset$, then (6.6) follows, and if not, then by (6.5) a_1 is adjacent to p vertices of M'. If $|M| \geqslant 2$, then (6.6) follows. Thus assume $1 \geqslant |M| \geqslant p \geqslant 1$. Since $(k - |M|)/2 \geqslant (5 - 1)/2 = 2$, for some $1 \leqslant i < j \leqslant k$,

$$\{e_i, e_j\} \subseteq \partial_H(u) - \partial_H(u, a_5),$$

and K has a four cycle C such that $|V(C)| = |E(C)| = 4$ and $\{e_i, e_j\} \subseteq E(C)$. Hence (6.7) follows.

By Lemma 6.2(2) we may assume that for each two vertices of T_K, $K - M'$ has an elemental star containing them. Set $a_0 = v$, and for $i, j = 0, 2, 3, 4$, set

$$p_{i,j} = |\partial_K(a_i, a_j)|,$$
$$r_i = |\{x \in V(K) - T_K \mid N_K(x) = T_K - a_i\}|.$$

For $i, j = 0, 2, 3, 4$, since $|\partial_K(\{a_i, a_j\})| \geqslant k$,

$$p_{i,j} \leqslant (k - 1)/2.$$

If a_1 is adjacent to a vertex of M' in G, then (6.6) follows. If for some $x \in V(G) - T$, $N_G(x) = \{a_1, a_i, a_4\}$ $(i = 2 \text{ or } 3)$, then (6.6) follows. Thus and by (6.5) we may assume that

$$|\partial_G(a_1, a_4)| = p.$$

If $a_4 \in Y$, then (6.6) easily follows, and thus let $T_H - Y = \{a_0, a_4\}$. Since $p_{0,4} \geqslant |\partial_G(a_1, a_4)| = p > 0$, by Lemma 6.2(1) we have

$$p_{4,2} > 0, \qquad p_{4,3} > 0, \qquad \text{or} \qquad r_0 > 0,$$

and

$$p_{0,2} > 0, \qquad p_{0,3} > 0, \qquad \text{or} \qquad r_4 > 0.$$

If $r_0 > 0$, $r_4 > 0$, $p_{0,2} \cdot p_{3,4} > 0$, or $p_{0,3} \cdot p_{2,4} > 0$, then (6.6) follows (note that K_i is $G(p_i, q_i)$ for $i = 1, 2$). Thus we may assume that

(6.8) $p_{0,2} > 0$, $p_{2,4} > 0$ *and* $r_0 = r_4 = p_{0,3} = p_{3,4} = 0$.

Assume $|M| = 0$. Then

$$d_G(a_3) = p_{2,3} + r_2 \qquad \text{and} \qquad p_{2,3} \leqslant (k-1)/2,$$

and so

(6.9) $r_2 \geqslant (k+1)/2 \geqslant p + 1$.

By comparing $d_G(a_2)$ with $d_G(a_4)$ we have

$$p_{0,2} + p_{2,3} = p_{0,4} + r_2.$$

Thus

(6.10) $p_{0,2} > p_{0,4} \geqslant p$.

From (6.9) and (6.10), (6.7) follows.

Now we may let $|M| > 0$. Since $\{a_2, a_3\} \subseteq Y$, we have

$$|\partial_K(Y)| = k = d_K(a_2) + d_K(a_3) - 2p_{2,3} - |M| = 2k - 2p_{2,3} - |M|,$$

and so

$$2p_{2,3} + |M| = k.$$

Since $d_G(a_3) = k = p_{2,3} + r_2 + |M|$,

$$r_2 = p_{2,3}.$$

Since $d_G(a_3) = 2r_2 + |M|$, $d_G(a_4) = p_{0,4} + p_{2,4} + r_2 + r_3 + |M|$, and $p_{2,4} > 0$ (by (6.8)), we have

(6.11) $r_2 \geqslant a_{0,4} + 1 \geqslant p + 1$.

By comparing $d_G(a_2)$ with $d_G(a_4)$, we have

$$p_{0,2}=p_{0,4}.$$

Thus

(6.12) $p_{0,2}+|M|\geqslant p+1.$

From (6.11) and (6.12), (6.7) follows.

Subcase 1.2.2. K has no nontrivial k-cut separating T_K.

We may assume that an end of each edge of K in T_K and K is elemental for T_K. The proof is similar as the case $|M|=0$ in the proof of Subcase 1.2.1.

Subcase 1.3. $\{a_2, a_m\}\subseteq V(G)-X$ and $|X\cap T|=3$.

Now $m=3$. By (6.4) K is $G(p', q')$. Set $X\cap T=\{a_1, a_4, a_5\}$. If H has a nontrivial k-cut $\partial_H(Y)$ $(Y\subseteq V(H)-u)$ separating T_H, then we may let $|Y\cap T_H|=2$. Then for Y or $V(G)-Y$ instead of X, Subcase 1.1 or 1.2 occurs. Thus we may assume that this is not the case and H is elemental for T_H. If either (6.13) or (6.14) holds, then the result follows:

(6.13) *For some* $e_i\in\partial_K(v)-\bigcup_{i=2}^3\partial_K(v, a_i)$, *$H$ has edge-disjoint paths* $P_1[a_1, u]$ *and* $P_2[a_1, u]$ *such that* $e_i\in E(P_1)\cup E(P_2)$ *and* $T_H\in\Gamma(H-\bigcup_{i=1}^2 E(P_i), k-2)$.

(6.14) *For* $l=2$ *or* 3 *and for some* $e_i\in\partial_K(v, x_l)$ *and* $e_j\in\partial_K(v)-\partial_K(v, x_l)$, *$H$ has edge-disjoint paths* $P_1[a_1, u]$ *and* $P_2[a_1, u]$ *such that*

$$\{e_i, e_j\}\subseteq E(P_1)\cup E(P_2) \qquad \textit{and} \qquad T_H\in\Gamma\left(H-\bigcup_{i=1}^{2} E(P_i), k-2\right).$$

Set $a_0=u$ *and for* $i, j=0, 1, 4, 5$, *set*

$$p_{i,j}=|\partial_H(a_i, a_j)|,$$
$$R_i=\{x\in V(H)-T_H\mid N_H(x)=T_H-a_i\},$$
$$r_i=|R_i|.$$

By (6.5) $p_{0,1}=0$.

Assume $p_{1,4}=p_{1,5}=0$. If $r_0\leqslant(k-1)/2$, then

$$r_4+r_5=d_G(a_1)-r_0\geqslant(k+1)/2\geqslant p'+1,$$

and so (6.13) or (6.14) follows. Thus let $r_0\geqslant(k+1)/2$. Since $d_G(a_0)=p_{0,4}+p_{0,5}+r_1+r_4+r_5$ and $d_G(a_5)=p_{0,5}+p_{4,5}+r_0+r_1+r_4$, we have

$$p_{0,4}+r_5=p_{4,5}+r_0.$$

Hence

$$d_G(a_4) = k \geqslant p_{0,4} + r_0 + r_5 \geqslant 2r_0 > k,$$

a contradiction.

Now we may let $p_{1,i} > 0$ for $i = 4$ or 5, say $i = 4$. Since $p_{0,1} = 0$ and by Lemma 6.2(3), we have

$$r_4 + r_5 \geqslant 2.$$

For each $x \in R_4 \cup R_5$, if x is adjacent to a vertex of $V(K) - T_K$ in G, then (6.13) follows, thus assume that $\partial_G(x, a_i) \neq \varnothing$ $(i = 2$ or $3)$. For each $x, y \in R_4 \cup R_5$, if $\partial_G(x, a_2) \neq \varnothing$ and $\partial_G(y, a_3) \neq \varnothing$, then (6.14) follows, thus assume that for $i = 2$ or 3, $\partial_G(x, a_i) = \partial_G(y, a_i) = \varnothing$, say $i = 3$, and that $r_4 + r_5 \leqslant p'$.

Assume $r_4 > 0$. For some $e_i \in \partial_K(v) - \partial_K(v, a_2)$, e_i is incident to a_4 or a vertex of R_1 in G, because

$$p' + q' \geqslant (k+1)/2 > p_{0,5}.$$

Thus (6.14) follows.

Now we may assume that $r_4 = 0$, $r_5 > 0$, and $p_{1,5} = 0$. Thus $p_{0,1} = p_{1,5} = r_4 = 0$, contrary to Lemma 6.2(1).

Subcase 1.4. $a_2 \in X$ and $a_m \in V(G) - X$.

Now $m = 3$.

Subcase 1.4.1. $|X \cap T| = 2$.

By (6.4) $H = G(p, q)$, and by (6.5) $p = 0$. Since $|T_K| \leqslant 4$, by induction K has a path $P[v, a_3]$ such that $T_K \in \Gamma(K - E(P), k-1)$. Let $e_1 \in E(P)$. H has an elemental star S_1 containing a_1 and e_1. Let S_2 be another elemental star of H. Then $T_H \in \Gamma(H - \bigcup_{i=1}^{2} E(S_i), k-2)$, and so the result follows.

Subcase 1.4.2. $|X \cap T| = 3$ and $|T - X| = 2$.

Assume that H has a nontrivial k-cut $\partial_H(Y) = \{f_1, ..., f_k\}$ $(Y \subseteq V(H) - u)$ separating T_H. Then we may assume that $|Y \cap T_H| = 2$, $a_2 \in Y$ and $a_1 \in X - Y$. Let H_1 (H_2) be the graph obtained from H by contracting $V(H) - Y$ (Y) to a new vertex u_1 (u_2). Then similarly as (6.4) H_i is $G(p_i, q_i)$ for some integers p_i and q_i $(i = 1, 2)$. If $p_2 = 0$, then the result easily follows. If $p_2 > 0$, then we may let $\{f_1, e_1\} \subseteq \partial_G(a_1)$ and we can easily deduce the result.

Now we may assume that H has no nontrivial k-cut separating T_H and H is elemental for T_H. Set $X \cap T = \{a_1, a_2, u, a_4\}$ and $T - X = \{a_3, a_5\}$. For a_1, a_2, u, a_4 instead of x_1, x_2, x_3, x_4, (a) or (b) of Lemma 6.2(3) holds. If (a) holds, then the result easily follows, thus assume that (b) holds. Since $|\partial_H(u) - \partial_H(u, a_2)| \geqslant (k+1)/2$ and $p' + q' \geqslant (k+1)/2$, for some $1 \leqslant i \leqslant k$,

$$e_i \in \partial_H(u) - \partial_H(u, a_2) \qquad \text{and} \qquad e_i \in \partial_K(v) - \partial_K(v, a_5),$$

and so the result follows.

Case 2. G has no nontrivial k-cut separating T.

We may assume that G is elemental for T. If $|T|=4$, then by Lemma 6.1 the result follows. Thus let $|T|=5$ and $m=3$. Set $T=\{a_1, a_2, a_3, a_4, a_5\}$ and for $1 \leqslant i, j, l \leqslant 5$, set

$$\begin{aligned} p_{i,j} &= |\partial_G(a_i, a_j)|, \\ R(i,j,l) &= \{x \in V(G)-T \mid N_G(x)=\{a_i, a_j, a_l\}\}, \\ r(i,j,l) &= |R(i,j,l)|. \end{aligned}$$

We require

(6.15) *For each distinct* $1 \leqslant i,j,l \leqslant 5$, G *has an elemental star containing* $\{a_i, a_j\}$ *or* $\{a_i, a_l\}$.

Proof. Assume that each elemental star of G does not contain $\{a_1, a_2\}$ nor $\{a_1, a_3\}$. Then

$$d_G(a_1) = p_{1,4} + p_{1,5} + r(1,4,5).$$

Since $p_{i,j} \leqslant (k-1)/2$ for each i,j, we have $r(1,4,5)>0$. Let F be the cut of G separating $\{a_1, a_4, a_5\} \cup R(1,4,5)$ from the rest of the graph, then

$$|F| \leqslant d_G(a_4) + d_G(a_5) - (p_{1,4} + p_{1,5} + 2r(1,4,5)) < k,$$

a contradiction. Now (6.15) is proved.

We return to the proof of Theorem 5. By (6.5)

$$p_{1,2} = p_{1,3} = r(1,2,3) = 0.$$

If $r(1,2,i)>0$ and $r(1,3,j)>0$ $(i,j=4$ or $5)$, then the result follows. Thus and by (6.15) we may assume that

$$r(1,2,4)>0 \qquad \text{and} \qquad r(1,3,i)=0 \quad (i=4,5).$$

By (6.15)

$$p_{i,5} + r(i,5,2) + r(i,5,4) > 0 \qquad (i=1,3).$$

If $p_{1,5}>0$, $p_{3,5}>0$, $r(1,5,2) \cdot r(3,5,4)>0$, or $r(1,5,4) \cdot r(3,5,2)>0$, then by Lemma 6.1 the result follows. Thus we may assume that for $(i,j)=(2,4)$ or $(4,2)$,

$$p_{1,5} = p_{3,5} = 0, \qquad r(1,5,i) = r(3,5,i) = 0,$$

and

$$r(1, 5, j) \cdot r(3, 5, j) > 0.$$

Assume $r(1, 5, 2) = r(3, 5, 2) = 0$. Then

$$d_G(x_1) = p_{1,4} + r(1, 2, 4) + r(1, 4, 5),$$

and

$$d_G(x_4) \geqslant p_{1,4} + r(1, 2, 4) + r(1, 4, 5) + r(3, 4, 5) > k,$$

a contradiction. Thus

$$r(1, 5, 4) = r(3, 5, 4) = 0.$$

Since $r(1, 2, 5) > 0$, by the same argument we have

$$p_{1,4} = p_{3,4} = 0.$$

Thus

$$d_G(x_1) = r(1, 2, 4) + r(1, 2, 5)$$

and

$$d_G(x_2) \geqslant r(1, 2, 4) + r(1, 2, 5) + r(2, 3, 5) > k,$$

a contradiction.

7. Proof of Theorem 6

Suppose that $k \geqslant 1$ is an integer, G is a graph, $T = \{s_1, ..., s_k, t_1, ..., t_k\} \subseteq V(G)$ and $T \in \Gamma(G, k)$. We prove that if $|T| = 3$, or if k is odd and $|T| = 4$ or 5, then (1.1) holds. We proceed by induction on k.

Assume $|T| = 3$. By Theorem 4 G has a path $p[s_k, s_k]$ such that $T \in \Gamma(G - E(P), k - 1)$. By induction for $k - 1$, (1.1) holds in $G - E(P)$, and so for k, (1.1) holds in G.

Assume that $k \geqslant 5$ is odd and $|T| = 4$ or 5. For some $1 \leqslant i < j \leqslant k$, if $|T| = 4$, then

$$s_i = s_j \text{ or } t_j,$$

and if $|T| = 5$, then

$$s_i = s_j \quad \text{or} \quad t_j \quad \text{and} \quad \{s_i, t_i\} \neq \{s_j, t_j\},$$

say for $i = k - 1$ and $j = k$. By Theorem 5 G has edge-disjoint paths $P_1[s_{k-1}, t_{k-1}]$ and $P_2[s_k, t_k]$ such that $T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-2)$. By induction for $k-2$, (1.1) holds in $G - \bigcup_{i=1}^{2} E(P_i)$, and so for k, (1.1) holds in G.

Thus for each integer $k \geqslant 1$,

$$\lambda'(k, 3) = \lambda(k, 3) = k,$$

and for each odd integer $k \geqslant 1$,

$$\lambda'(k, 4) = \lambda'(k, 5) = k.$$

By Theorem 3 for each odd integer $k \geqslant 1$,

$$\lambda(k, 4) = \lambda(k, 5) = k \qquad \text{and} \qquad \lambda(k+1, 4) = \lambda(k+1, 5) = k+2.$$

Now Theorem 6 is proved.

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